Geometric support of algorithms for solving Problems of higher mathematics

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The need to improve the level of mathematical in particular geometric training of students of technical universities is due to modern technologies of computer-aided design. They are based on mathematical models of designed products, technological processes, etc., taking into account a large variety of source data. Therefore, from the first years of technical universities, when studying the cycle of mathematical disciplines, it is advisable to interpret a number of issues in terms and concepts of multidimensional geometry. At the same time, the combination of constructive (graphical) algorithms for solving problems in descriptive geometry with analytical algorithms in linear algebra and matanalysis allows us to summarize their advantages: the constructive approach provides the imagery inherent in engineering thinking, and the analytical approach provides the final result. The article shows the effectiveness of combining constructive and analytical algorithms for solving problems involving linear and nonlinear forms of many variables using specific examples.

Keywords: *descriptive geometry, linear algebra, multidimensional forms - linear and nonlinear, constructive and analytical solutions, geometric model*

1. Introduction

In the first years study of technical universities two approaches are considered when studying mathematical cycle disciplines:

- constructive (graphic) in the teaching of descriptive geometry;
- analytical with emphasis on the study of numerical algorithms (linear algebra, calculus).

Only in the course of analytical geometry are algebra and geometry considered together. At the same time, none of these courses even talk about multidimensional spaces, although they consider systems of linear equations from several unknowns, study methods for differentiating and integrating functions of many variables ets. Each of these approaches has its own advantages. If the constructive approach provides the imagery inherent in engineering thinking, then the analytical approach provides the final result. Therefore, their rational combination should contribute to the successful development of the course being studied. In this regard, this publication is devoted to the justification of making some additions to the course of descriptive geometry, which, in our opinion, will contribute to the geometric support of algorithms for solving a number of problems of higher mathematics.

2. Geometric representation of the solution of systems of linear equations

Let's start with linear algebra. In high school, students are taught to solve systems of two linear equations with two unknowns and three linear equations with three unknowns. Students understand the geometric meaning of the systems being solved. In the first case, they calculate the coordinates of the intersection point of two straight lines, and in the second – the coordinates of the common point of three planes. At the University, they study the solution of systems of four or more linear equations with the corresponding number of unknowns, using the Gauss algorithm. Unfortunately, now them do not explain the geometric meaning of a linear equation from many unknown ones, because the programs of existing courses in descriptive and analytical geometry are focused only on the study of linear and nonlinear forms of three-dimensional space.

This gap can be most easily and clearly eliminated by expanding the subject of descriptive geometry with the forms of four-dimensional space and generalizing the twocard drawing of Monge with the drawing of Radishchev (Fig. 1).

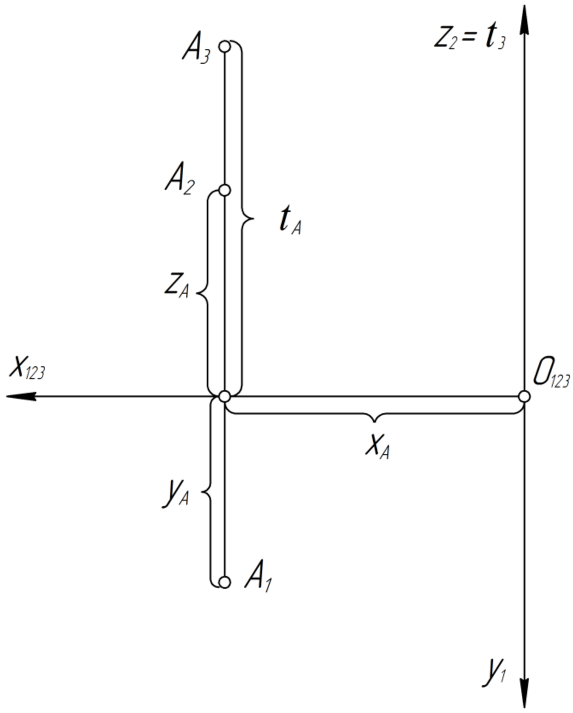

Fig. 1. Setting point A of a four-dimensional space in the Radishchev drawing

At the same time, it is logical and simple enough to graphical definition lines and planes of three-dimensional space ([1]: 2.1, 2.2) and generalize it to define linear forms of multidimensional space ([1]: 2.2.3). This fully applies to their analytical task ([1]: 2.3). If linear algebra courses do not provide a geometric interpretation of a rectangular matrix and its rank , this is now available ([1]: 2.3.2):

− a rectangular matrix consisting of *n-p* rows and *n* + 1 columns defines a *p*-plane defined by a system of *n-p* linear equations from *n* unknowns;
− rank *R* = *n-p*, where *p* is the dimension of the *p*-plane through which all the hyperplanes of this *n* - dimensional space pass.

Thus, the extension of the subject of descriptive geometry by multidimensional (at the first stage − four-dimensional) linear forms and their analytical assignment in the form of linear equations or systems of equations allows us to visually (geometrically figuratively) represent them as *p* - planes, their intersections and unions (enclosing spaces). As a result, the existence of a relationship between their methods and the rationalization of algorithms for solving certain problems is revealed.

To confirm this thesis, section 6.2.1 [1] provides two examples. In the first example, we discuss an algorithm for constructing the intersection point of three planes α, β, and γ. As a rule, in descriptive geometry, this problem is solved in this sequence:
- the line up of intersection of *l* planes α and β is constructed;
- the desired point *K* of the intersection of the line *l* and the plane γ is constructed.

In the language of linear algebra this problem is reduced to solving a system of three linear equations with three unknowns: by elementary transformations, the square matrix of coefficients is reduced to a trapezoidal one , which in descriptive geometry corresponds to the transformation of one plane into a projecting one. Therefore, a simpler calculation of the determinant of the transformed (trapezoidal) matrix corresponds to a simpler graphical way of constructing a common point of three planes, one of which is the projecting one.

The second example shows a graphical implementation of the Gauss method for sequentially reducing the dimension of the problem to be solved ([1], p. 6.2.1). On the example of a graphical solution to the problem of constructing a point *K* of the intersection of a line *l* with a hyperplane Σ^3 (*ABSD*), the drawing clearly shows a sequential decrease in the dimension 4→3→2→1.

Thus, the questions discussed above convincingly show the unity of the subject of linear algebra and multidimensional descriptive geometry, the usefulness of parallel solutions of geometric problems using graphical and analytical methods.

3. Kinematic method for forming multidimensional surfaces

Let's consider examples of geometric support for solving problems involving nonlinear forms. Therefore, we will first show the kinematic method of their formation, which is characterized by clarity and implements the principle of separation, which is widely used in computational mathematics.

In the *Oxy* coordinate plane, point *A*, moving according to some law, forms a curve α^1($y = f(x)$) (fig. 2). In turn, the curve α^1, moving in the space *Oxyz* by its law, forms ("sweeps") a two- dimensional surface α^2($z = \gamma(x, y)$). The surface α^2, moving in the four- dimensional space *Oxyzt* in the direction of the axis *Ot*, forms a 3-surface α^3 ($t = \varphi(x, y, z)$), which, in turn, "sweeps" the 4-surface α^4 (in fig.2 not shown). This process continues until the ($n-2$) - surface α^{n-2} "sweeps" the hypersurface α^{n-1}.

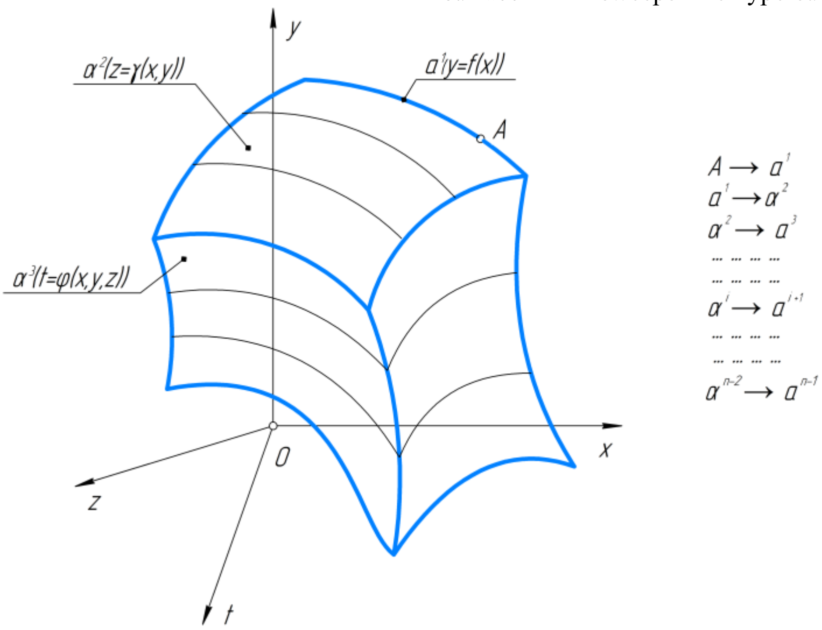

Fig. 2. Kinematic method for forming multidimensional surface

Thus, the hypersurface α^{n-1} is a one - parameter ∞^1 set of ($n-2$) - surfaces (stratified into a bundle of ($n-2$) - surfaces). In turn, the ($n-2$) - surface is stratified into a bundle of ($n-3$) - surfaces, etc. As a consequence, the membership problem is solved by constructing an ($n-1$) - dimensional nonlinear **flag** (see [1], p. 2. 2. 3).

The question arises, how to construct a tangent plane to the hypersurface being constructed?

In the course of mathematical analysis, the tangent *t* to the curve *m* at its point *M* is called the limit position MN^i of the secant *MN*, which it occupies when the point *N* along the curve *t* tends to the point *M*. In other words, a tangent *t* is such a secant (chord) that intersects the curve

m at two coinciding points $M = N^i$. This definition also applies to the touch of curved lines, flat and spatial. Since two curves can intersect at several points, two, three, or more points can coincide in the limit. Therefore, they talk about two-point, three-point, etc. touches. For example, two second-order curves can have two-point, three-point, and four-point touches. In the language of mathematical analysis, this means that in the case of a two-point touch, the coordinates of the coinciding points satisfy the equations of both curves, and the first derivatives taken from the equations of these curves are equal at this point. In the case of a three-point touch, the second derivatives are additionally equal, and in the case of a four-point touch, the third derivatives are also equal. In engineering practice, it is customary to call two-point, three-point, etc. touches, respectively, touches of the first, second, and n-th order of smoothness. Curves made up of arcs that touch curves are called outlines.

These concepts are applied in the appropriate interpretation to the touch of surfaces. As noted above, the construction of curves and surfaces is of great practical importance. In theory, they are generalized to touch in multidimensional spaces. In computational terms, this is reduced to operating with partial derivatives of functions of many variables.

Let's start by considering the theoretical provisions for constructing a tangent plane to a surface in three-dimensional space.

In differential geometry, it is shown that the set of tangents t^i drawn to a surface Φ at some point A belongs to the plane τ, if the point A is its regular (ordinary) point. If the point A is a special point of the surface Φ, then the set of tangents t^i forms a conic surface τ with a vertex at this point [9].

Since the tangent plane τ is uniquely defined by two straight lines, the algorithm for constructing it consists of the following steps:

- through this point A of the surface Φ, any two of its lines a, b are drawn;
- at point A, tangents t^1, t^2 are constructed to the selected lines a, b; intersecting lines t^1, t^2 define the plane τ, touching the surface Φ at point A.

This algorithm is the basis of an analytical method for constructing a tangent plane τ of a surface Φ at its point A. If the equation Φ $(x, y, z) = 0$ of the surface is substituted with the values $x = x_A$, $y = y_A$, $z = z_A$, then we get the equations of the sections a, b, with the surface Φ by planes passing through the point A and parallel to the coordinate planes Oyz, Oxz, Oxy, respectively. Partial derivatives

$$\frac{\partial\Phi(x,y,z)}{\partial x}, \frac{\partial\Phi(x,y,z)}{\partial y}, \frac{\partial\Phi(x,y,z)}{\partial z}$$

at point $A(x_A, y_A, z_A)$ are the angular coefficients of the tangents t_1, t_2, t_3, held at point A for curves a, b, c.

The equation of the tangent plane τ has the form:

$$\frac{\partial\Phi}{\partial x}(x - x_A) + \frac{\partial\Phi}{\partial y}(y - y_A) + \frac{\partial\Phi}{\partial z}(z - z_A) = 0$$

Thus, analytically, the construction of a tangent plane in three-dimensional space is reduced to the calculation of partial derivatives of functions Φ $(x, y, z) = 0$ from three variables.

Let's consider an example of constructing a tangent plane τ^2 to a surface Φ^2 in three-dimensional space. In textbooks on descriptive geometry, the construction of tangent planes to the simplest surfaces (sphere, cone, etc.) is given. On the surface at this point A we draw two graphically simple lines a and b.

The tangents t_a and t_b define the desired tangent plane τ∋ A. Since engineering surfaces are complex, the curves a and b take the surface sections as planes parallel to the coordinate planes of the projections. In our example, the tangent plane τ^2 is structurally defined by two tangents t_x^1, t_y^1, drawn to the sections g_x^1, g_y^1 of the given surface Φ^2 (Fig. 3).

Structurally, the method of stratification is used to solve such problems. The application of the bundle idea in solving problems involving nonlinear forms is shown in [8], in particular, by the example of calculating partial derivatives (p. 6. 2. 2). For example, in engineering practice, they are used when constructing the tangent plane τ of the surface Φ at its point A.

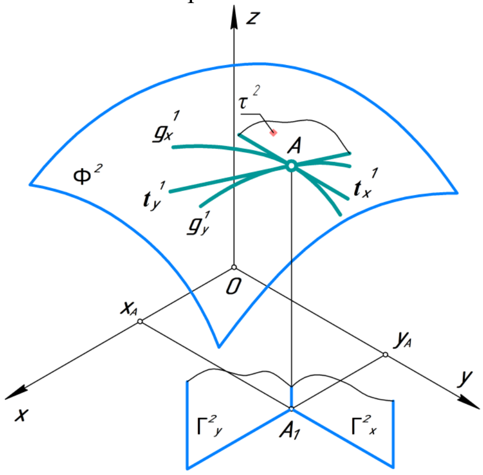

Fig. 3. Construction of the tangent plane $\boldsymbol{\tau^2}$ to the surface $\boldsymbol{\Phi^2}$

We generalize the solution of this problem to the construction of a tangent of the 3 - plane τ^3 to the 3-surface Φ^3 in four-dimensional space.

In the fig. 4 shows an example of generalization of the considered algorithm to the construction of a tangent of the 3 - plane τ^3 to the 3-surface Φ^3 in four-dimensional space.

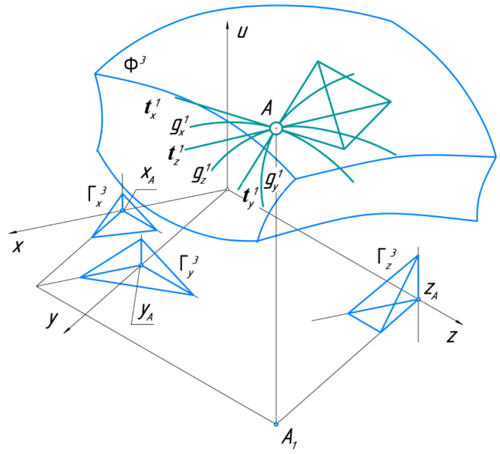

Fig. 4. Scheme for constructing a tangent of the 3-plane τ^3 to the 3-surface F3 in four-dimensional space

Let 3-the surface of Φ^3 be given explicitly by the equation:

$$u = f(x, y, z).$$

Using the three known coordinates x_A, y_A, z_A of a certain point A, we calculate its fourth coordinate u_A from this equation. Structurally, this is done by drawing three projecting 3-planes Γ_x^3 $(x = x_A)$, Γ_y^3 $(y = y_A)$, Γ_z^3 $(z = z_A)$. Each of them intersects this 3-surface Φ^3, respectively, on 2-surfaces (3 + 3 – 4 = 2): $g_x^2 = \Gamma_x^3 \cap \Phi^3$, $g_y^2 = \Gamma_y^3 \cap \Phi^3$ $g_z^2 = \Gamma_z^3 \cap \Phi^3$ (they are not shown in fig. 4).

These three 2-surfaces belonging to the 3-surface of Φ^3 intersect in pairs along three flat curves (2 + 2 – 3 = 1), belonging to 2-planes parallel to the corresponding coordinate planes:

$$g_x^1 = g_y^2 \cap g_z^2 \quad \big(u = f_1(x)\big),$$
$$g_y^1 = g_x^2 \cap g_z^2 \quad \big(u = f_2(y)\big),$$
$$g_z^1 = g_x^2 \cap g_y^2 \quad \big(u = f_3(z)\big).$$

It follows that $g_x^1 \parallel Oxu$, $g_y^1 \parallel Oyu$, $g_z^1 \parallel Ozu$. These three curves intersect at $A \in \Phi^3$. The angular coefficients of the tangent t_x^1, t_y^1, t_z^1, carried out at the point to these curves, the essence of partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$. Three tangents t_x^1, t_y^1, t_z^1 passing through point A and located in perpendicular planes define the desired 3-plane τ^3, tangent to this 3-surface Φ^3 at its point A. Subsequent generalizations of the algorithm are fairly obvious. Such generalizations of the above to higher-dimensional spaces explain the geometric meaning of partial derivatives of functions of n-variables.

4. Conclusions

The application of the bundle idea in solving problems involving nonlinear forms is shown in [1] using examples of calculating partial derivatives and definite integrals. Geometric interpretation of computational algorithms for solving these and a number of other problems involving nonlinear forms, in our opinion, should improve the quality and level of mathematical, in particular, geometric training of students of technical universities. This requirement is relevant in modern conditions, because the optimization of parameters of designed products, technological processes, etc. is based on their mathematical models, taking into account a large variety of source data.

Knowledge of the algorithm for solving the problem would allow the student to establish a close connection with other special disciplines at the stage of design, calculations and visualization of data in CAD systems.

The real implementation of departments engineering graphics of the concept of geometric support of algorithms for solving problems of higher mathematics is possible in the educational process only in the presence of highly qualified teachers who possess constructive (graphical) and analytical methods for solving geometric problems. Unfortunately, several generations of teachers of the Department of engineering graphics have considered and continue to consider descriptive geometry as a purely graphical discipline and analytical methods for solving problems do not apply. Therefore, the actual task of the scientific and methodological Council is to organize an effective system of professional development of teachers.

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